Chaotic versus random ionization

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We use Kepler maps to study the classical ionization of the hydrogen atom driven by electric fields; we compare the ionization rate (IR) due to chaotic motion (constant field amplitude) with the IR due to randomness (random field amplitude). We find that for weak fields the ionization due to random amplitude is more effective, while for strong fields the ionization due to chaos is more effective; so, there is a cross of such ionization curves. A physical explanation of this phenomenon, based on the trajectory behavior, is given.

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It is known that generally a random component in the driven force destabilizes physical systems [1–4]. Another source of destabilization can be the chaotic behavior in time dependent systems [5–8]. Recall that the chaotic behavior has been described as a kind of instability generated by deterministic dynamics, so randomness and chaoticity are expected to be somehow related. In fact, the principles of ergodic theory do not distinguish both kinds of systems, the main point being the presence of invariant measures with respect to the dynamics; as an important example consider Oseledec's multiplicative ergodic theorem [9–11] (and its variations) invoked to rigorously define Lyapunov exponents for both random and chaotic systems. In some calculations random variables have successfully mimicked chaotic orbits [12,13].

Usually the Hamiltonian chaotic dynamics resemble random walks in phase space, but with the presence of stability islands. In this Brief Report we address a very simple and interesting question: randomness or chaoticity is more effective in destabilizing a physical system? Certainly the answer should depend on many different variables, so we need to be very definite on our choices of the system and underlying conditions. We choose a classical driven atomic system so that (i) we can clearly monitor the destabilization through atomic ionization, which, in principle, could be tested in experiments; (ii) it is possible to reduce the relevant dynamics to maps, speeding up considerably the numerical calculations; (iii) the original classical system presents chaos for suitable intensities of the driven force; (iv) it's very easy to introduce some randomness in the force intensity; and, (v) finally, we get an immediate comparison between both kind of motion via the potential intensity.

Some comments are in order. Item (ii) needs some justification, so some specific numerical calculations, without the map approximation, were performed on our system (see below). In the case of random force (iv) one needs to be sure the main contribution for ionization is due to the randomness itself; we shall check this by numerically computing a diffusion constant as function of the force intensity. Now we discuss our system and also detail how we have controlled the points just mentioned.

The one-dimensional (1D) hydrogen atom under intense periodic electric field is an important chaotic physical system; it is amenable to experiments [14] and its theoretical study [5,15–21] can be reduced to the Kepler map [17,18], which cleverly reduces the dynamics to the perihelium. Although it is a great simplification over the full Hamiltonian, it has provided a good description of the whole system. The Hamiltonian of this system under a monochromatic field with amplitude ϵ and frequency ω is

$$H(x,t)=(1/2)p_x^2-1/|x|+\epsilon x\sin(\omega t), \tag{1}$$

and the corresponding (usual) Kepler map [17,18],

$$\begin{aligned} N_{k+1}&=N_k+0.822\pi\epsilon\omega^{-5/3}\sin(\phi_k),\\ \phi_{k+1}&=\phi_k+2\pi\omega(-2\omega N_{k+1})^{-3/2}, \end{aligned} \tag{2}$$

where N is the classical equivalent of the number of absorbed photons and ϕ is the phase of the electric field at the perihelium of the electron orbit (k counts the number of times the electron reaches the perihelium). We shall also denote by n the classical quantity equivalent to the principal quantum number.

We introduce the randomness in Eq. (1) by taking $\epsilon=\epsilon(t)$ assuming randomly $\pm\overline{\epsilon}$, and each value being constant during a Kepler period, i.e., it does not change in the time interval $\Delta t=2\pi/\omega_K$ (ω_K denotes the Kepler frequency). In this way the only modification of Eq. (2) is the consideration of $\epsilon=\pm\overline{\epsilon}$ according to a random sequence—we took ϵ equally distributed between these two values—and we shall call it random Kepler map.

The use of the Kepler period in randomly changing field intensity is pragmatical. First, the usual Kepler map itself is derived taking into account that the main contribution of the electric field occurs at the perihelium, so that it evolves the system for a Kepler period. Then our random choice is quite natural. Second, such choice simplifies enormously the numerical calculations. We have checked, for some parameter values, that the Kepler map (2) actually reproduces the general behavior of the solutions obtained from direct integration of the Hamiltonian equations derived from Eq. (1), as discussed later on.

It is well known that systems (1) and (2), with suitable constant $\epsilon=+\overline{\epsilon}$ (or $\epsilon=-\overline{\epsilon}$), give rise to chaotic motion and ionization [18,8]. Our random modification also generates ionization. Therefore, we use both Kepler maps to calculate

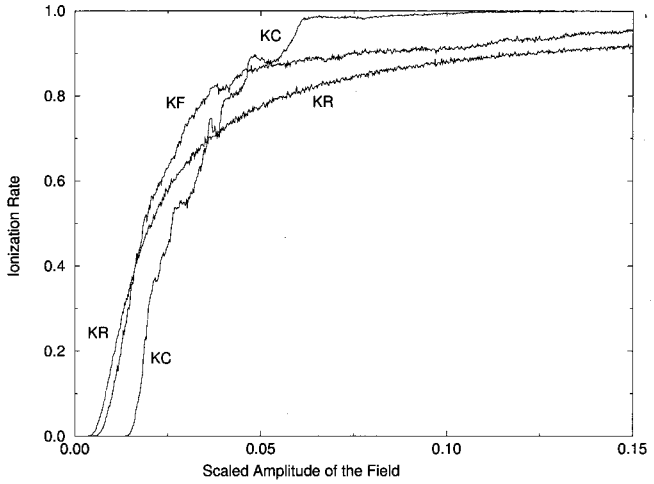

FIG. 1. Ionization rate (whose maximum is one) as a function of the scaled field amplitude $\epsilon_0=\bar{\epsilon}n^4$ in atomic units (a.u.) for fixed scaled frequency $\omega_0=\omega n^3=3.0$ a.u. The curves (KC), (KF), and (KR) are obtained by iteration of the Kepler maps with the constant, Fibonacci and random field amplitude sequences, respectively. It was used 1×10^4 initial conditions for each value of ϵ_0.

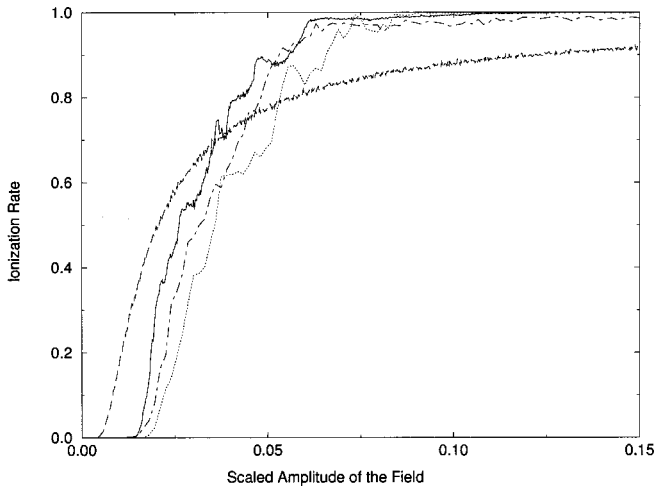

FIG. 2. Ionization rate (whose maximum is one) as functions of the scaled field amplitude in a.u. for $\omega_0=3.0$ a.u. The curves were obtained with the constant sequence for the Kepler map (solid) and the Hamiltonian equations (dotted), and also with the random sequence for the Kepler map (dashed) and the Hamiltonian (dot dashed). It was used 1×10^3 initial conditions for the Hamiltonian equations and 1×10^4 for the Kepler maps.

the ionization rate (IR) as a function of the scaled field amplitude $\epsilon_0=\bar{\epsilon}n^4$, and then compare the random and chaotic ionizations.

We have found that the IR for low field intensity, with the random sequence to shape its amplitude, is larger than that generated by the usual Kepler map (constant sequence); see Fig. 1. This is expected since the usual Kepler map preserves many stability islands for low field intensity, and most trajectories are restricted to Kolmogorov-Arnold-Maser (KAM) tori. The random case breaks all stability islands for any nonzero value of the field amplitude $\bar{\epsilon}$, and eventually many trajectories escape.

The result is reversed for the IR under high field intensity, i.e., in this case the ionization is more effective for the chaotic Kepler map with constant amplitude sequences than for the corresponding map with random sequences; see Fig. 1. We checked that this occurs because in the random case there are no stability islands that prevent the decreasing of the quasienergy ($E_0=\omega N n^2$) indefinitely. The decreasing of the quasienergy in the random case is connected to the decreasing of the radii $r=|x|$ of some electron trajectories. The approximation of the electron to the nucleus is a kind of classical localization in the random case, which is absent in the usual Kepler map. In fact, as it is well known, even for high intensity fields the usual Kepler map preserves some stability islands and $E_0=\omega N n^2$ cannot decrease indefinitely. However, the phase space is virtually homogeneous for the random case, i.e., the trajectories are dense in phase space.

In the regime of the high field intensity we have found that the system is strongly perturbed and each trajectory evolves into one of two possible asymptotical limits (except, of course, the trajectories restricted to the preserved tori): the limit $r\to\infty$ (ionization) or the limit $r\to0$. If there are no stability islands, the two limits are possible; if there are islands near the origin only the limit $r\to\infty$ is possible for the trajectories of the Kepler map. In the random case the trajectories are distributed between both limits, while for the usual Kepler map (with weak or intense field) only the $r\to\infty$ is possible; this is the mechanism responsible for the crossing of the IR curves, also justifying why for intense fields the Kepler map with constant sequence presents larger IR than the random case (see Fig. 1).

For some parameter values we have also calculated the IR by integration of the differential equations from the classical Hamiltonian (1), and then compared such results with those obtained with the Kepler maps (see Fig. 2). Our intention was to check whether the found ionization crossings, illustrated in Fig. 1, were an artifact of the Kepler map approximations. We have considered the Hamiltonian with constant field amplitude and also with its sign randomly changing at the perihelium. For both cases the Hamiltonian and Kepler map present similar island structures in phase space; also the values of IR are comparable in each case, so that the order inversion of the IR curves as function of the field intensity is also present for the Hamiltonian equations. For the random Hamiltonian we have checked that similar results are obtained even though the sign of the field changes more frequently along the electron trajectory, i.e., near the perihelium and also in other positions of its trajectory. Such results corroborate the use of above Kepler map approximation for random fields. We are thus justified in using Kepler maps in the simulations [see (ii) at the begining of this work]. Notice, however, that the effects are more pronounced in the calculations based on the Kepler maps (see Fig. 2). One reason for such quantitative difference is that the Kepler map with constant sequence is in good agreement with the Hamiltonian, while for nonconstant sequences such agreement is not so good; perhaps for continuous models the phase space island structure breaks hardly by sequence perturbations in comparison to the respective discrete maps.

We have calculated the averages $\langle N^2\rangle$ and $\langle n^2\rangle$ (over ini-

tial conditions) as functions of time for both the usual and random Kepler maps. In the case of the usual Kepler map no simple law was found for the time dependence of $\langle N^2\rangle$, even for large values of $\overline{\epsilon}$; recall that for small values of $\overline{\epsilon}$ the Kepler map is nearly integrable. On the other hand, $\langle N^2\rangle$ for the random Kepler map is diffusive for all values in the range of amplitudes we have considered (i.e., $2\times10^{-4}\leqslant\overline{\epsilon}\leqslant0.2$), $\langle N^2\rangle(t)\approx D(\overline{\epsilon})t$, and the dependence of the diffusion constant with the amplitude of the field being $D(\overline{\epsilon})\sim\overline{\epsilon}^{\delta}$, with $\delta=2.14$ fitting very well in such range of amplitudes. This strongly indicates that the random component is indeed the main ingredient responsible for the ionization in the random Kepler map for any nonzero $\overline{\epsilon}$ value. We have also found that, discarding the trajectories as soon as they are ionized, $\langle n^2\rangle$ decreases monotonically; since $\langle n^2\rangle$ is proportional to the radius of the electron orbit, we have a quantitative evidence that, in the case of random Kepler map, the trajectories that do not ionize penetrate into the atom (i.e., $r\rightarrow0$), while for the usual Kepler map this phenomenon is forbidden due to the presence of stability islands in phase space.

We have also considered an almost periodic driving, with the sign of the field intensity $\pm\overline{\epsilon}$ in the Kepler map following the Fibonacci [22,23] sequence, which is obtained from the replacement rules

$$+\rightarrow+-\quad\text{and}\quad-\rightarrow+;$$

beginning with ''+'' one uses the above rules in concatenation to get

$$+-++-+-++-++-\cdots$$

In the Fibonacci case we have, in fact, got intermediate values of the IR between the other two cases we have analyzed—see Fig. 1. Notice the inversion of the IR effectiveness with respect to the other two cases.

To conclude we try to put our results in a more general perspective, although based on just one physical system. We have found that for small perturbation intensity the chaotic behavior fills a small part of phase space and the destabilization is less effective than the random one. There is a crossover and for large perturbation the random destabilization becomes less effective than the chaotic one. Intermediary behavior is possible, as we have found by using an almost periodic driven case.

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